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OPERATOR PRODUCT EXPANSION AND FACTORIZATION IN THE H_3^+ -WZNW MODEL

J. TESCHNER

ABSTRACT. Precise descriptions are given for the operator product expansion of generic primary fields as well as the factorization of four point functions as sum over intermediate states. The conjecture underlying the recent derivation of the space-time current algebra for string theory on ADS_3 by Kutasov and Seiberg is thereby verified. The roles of microscopic and macroscopic states are further clarified. The present work provides the conformal field theory prerequisites for a future study of factorization of amplitudes for string theory on ADS_3 as well as operator product expansion in the corresponding conformal field theory on the boundary.

1. INTRODUCTION

The H_3^+ -WZNW model can be used to study string theory on backgrounds that contain an ADS_3 -part. Interest in such backgrounds was renewed¹ by Maldacena's conjectures [2], which predict that string theory on backgrounds with ADS_3 should be equivalent to a two dimensional conformal field theory on the boundary of ADS_3 . One point that makes the study of the ADS_3 case of Maldacena's conjectures particularly interesting is that the exact solvability of the H_3^+ -WZNW model [3][4] makes it possible to study the ADS-CFT correspondence beyond the supergravity approximation [5][6]. More precisely, it becomes possible to investigate the regime where the string coupling is small, but where the string length may be of the same order as the curvature radius of ADS_3 , which is a genuinely "stringy" regime.

Important steps in this direction were taken in the already mentioned papers [5][6] by showing (among others) that the spectrum generating algebra for string theory on ADS_3 (which is identified with the chiral algebra of the corresponding CFT on the boundary) can be constructed by worldsheet methods. This was first done by free field methods in [5] which is good enough for the identification of observables and their transformation properties, as well as the study of sectors describing strings stretched along the boundary (as discussed in [6][7]). In general one needs to consider the full interacting H_3^+ -WZNW model as was done in [6], where the spectrum generating algebra was constructed in terms of primary fields and currents of the H_3^+ -WZNW model. This construction was based on a conjecture (eqn. (2.35) of [6]) on the OPE of a certain H_3^+ -WZNW primary field Φ_1 with a general primary field. One result of the present paper will be to prove this conjecture on the basis of the results of [4]. The crucial point is that the contributions to the OPE of

¹Earlier investigations had studied mainly the physical spectrum via the $SU(1,1)$ -WZNW model. See e.g. [1]

two primary field with OPE coefficients that contain delta-functions follow unambiguously from the “smooth” coefficients determined in [4] via analytic continuation.

On the other hand, the author would like to take the opportunity to complete the results of [4] in view of future applications such as to string theory on ADS_3 . This includes choosing a new normalization for the primary field which is more natural from the point of view of string theory on ADS_3 than the one used in [4], and describing how the operator product expansion (OPE) of two general primary fields is fully and unambiguously determined from the results in [4].

One of the main objectives of the present work however is to present a discussion of factorization for the four point function in the H_3^+ -WZNW model which should provide all ingredients needed from the H_3^+ -WZNW model for a future study of factorization and OPE in the *spacetime* CFT. This includes a clarification of the roles of macroscopic and microscopic states and their correspondence to operators. It is sometimes claimed that state-operator correspondence breaks down in the H_3^+ -WZNW model. This turns out to be rather misleading in view of the results of the present paper. It is found that the distinction between macroscopic and macroscopic states is in fact inessential as far as state-operator correspondence is concerned. This will basically be a consequence of the remarkable analyticity properties of the three point function found in [4]. The difference between microscopic and macroscopic states *is* of course essential for their appearance as intermediate states in correlation functions.

The paper is organized as follows: Section 2 summarizes results from [4], rewritten using the new normalization of primary fields. The motivation for choosing this normalization will be explained. Besides that, the main new point of this section is a more precise discussion of macroscopic and microscopic states.

Section 3 describes how to obtain the OPE in the general case from the special case presented in Section 2 via analytic continuation. It is found that the OPE generically contains contributions which involve delta-functions in the OPE coefficient. The conjecture used in [6] is a particular case of the OPE’s studied.

The fourth section then discusses factorization of the four point function. It takes the form of an integral w.r.t. the label j of the intermediate representation, where the integrand factorizes into structure constants, two-point function and conformal blocks.

The author has chosen to defer a couple of more technical aspects to appendices in order to facilitate access to the main results and ideas in the body of the paper.

Appendix A provides some details on how to fix the normalization and the relation between the normalizations used here and in [4].

Appendix B demonstrates how the contributions to OPE and factorization of four point function are determined by the current algebra symmetry. The infinite-dimensionality of the zero mode representation introduces some new features as compared to other CFT. The results of this appendix, although rather technical in nature, are of fundamental importance for the bootstrap approach to the H_3^+ -WZNW model.

Appendix C finally discusses the Knizhnik-Zamolodchikov (KZ) equations that correlation function of the H_3^+ -WZNW model satisfy.

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2. SOME RESULTS ON THE H_3^+ -WZNW MODEL

The H_3^+ -WZNW is a conformal field theory that may be described by the Lagrangian

$$(1) \quad L = k(\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}),$$

see [3] for more details on the Lagrangian formulation of this model. The aim of the present section is to briefly summarize the picture that has emerged from [3][4][8]. Details on the relation of the normalizations used here and in [4] are given in Appendix A.

2.1. Canonical quantization. Canonical quantization is performed by introducing canonical momenta $(\Pi, \beta, \bar{\beta})$ for the fields $(\phi, \gamma, \bar{\gamma})$ and defining the operator algebra by canonical commutation relations. The space of states \mathcal{V} is then defined as tensor product of a Schrödinger representation for the zero modes $(\phi_0, \gamma_0, \bar{\gamma}_0)$ on $L^2(H_3^+, e^{2\phi_0} d\phi_0 d^2\gamma_0)$ with a representation of the non-zero modes on a Fock-space \mathcal{F} , see [4] for more details.²

States are represented by wave-functions $\Psi(h) \equiv \Psi(\phi_0, \gamma_0, \bar{\gamma}_0)$ taking values in \mathcal{F} . There is a nondegenerate hermitian form on \mathcal{F} that makes the Hamiltonian symmetric, but which is not positive definite.

The space of states \mathcal{V} carries a representation of two (left/right) commuting isomorphic \mathfrak{sl}_2 current algebras generated by modes J_n^a and \bar{J}_n^a , $a = +, 0, -$. The modes J_n^a satisfy the following commutation relations

$$(2) \quad \begin{aligned} [J_n^0, J_m^0] &= -\frac{k}{2}n\delta_{n+m,0} & [J_n^-, J_m^+] &= 2J_{n+m}^0 + kn\delta_{n+m,0}. \\ [J_n^0, J_m^\pm] &= \pm J_{n+m}^\pm \end{aligned}$$

The expressions for the generators in terms of the canonical fields contains terms proportional to $e^{-2\phi_0}$ which reflect the interaction of the H_3^+ -WZNW model. These terms disappear for $\phi_0 \rightarrow \infty$, in which case one recovers free field representations of the \mathfrak{sl}_2 current algebra. These free field representations differ from the usual free field representations by a different treatment of the zero modes: The zero mode representation space is $\mathcal{S}(\mathbb{C})$ and the generators $\gamma_0, \bar{\gamma}_0$ act as multiplication operators, their conjugate momenta $\beta_0, \bar{\beta}_0$ as the corresponding derivatives.

As usual one has associated with the current algebras two commuting Virasoro algebras with generators L_n, \bar{L}_n by means of the Sugawara construction:

$$(3) \quad L_m = \frac{1}{2(k-2)} \sum_{k \in \mathbb{Z}} : -2J_{-k}^0 J_{m+k}^0 + J_{-k}^+ J_{m+k}^- + J_{-k}^- J_{m+k}^+ : \\ \text{where} \quad : J_m^a J_n^b := \begin{cases} J_m^a J_n^b & \text{if } m < n \\ \frac{1}{2}(J_n^a J_m^b + J_m^b J_n^a) & \text{if } m = n \\ J_n^b J_m^a & \text{if } m > n, \end{cases}$$

²In comparison to [4] it should be noted that the fields $(\phi, \gamma, \bar{\gamma})$ and $(\Pi, \beta, \bar{\beta})$ used here correspond in [4] to $(-b\varphi, bv, b\bar{v})$ and $(b\Pi_\varphi, b\Pi_v, \Pi_{\bar{v}})$ respectively, where $b^{-2} = k - 2$. Moreover, the central charge k used here is the negative of the central charge that appears there.

and correspondingly for the \bar{L}_n .

2.2. Spectrum. The decomposition of \mathcal{V} into irreducible representations [3] can be written as

$$(4) \quad \mathcal{V} \simeq \int_{\mathcal{C}^+}^{\oplus} dj \, \mathcal{R}_j,$$

where $\mathcal{C}^+ = -\frac{1}{2} + i\mathbb{R}^+$ and the representations \mathcal{R}_j are defined as follows: One starts with a representation P_j for the zero modes J_0^a, \bar{J}_0^a which corresponds to a principal series representation of $SL(2, \mathbb{C})$. It may be realized e.g. on the Schwartz-space $\mathcal{S}(\mathbb{C})$ of functions on \mathbb{C} by means of the differential operators \mathcal{D}_j^a

$$(5) \quad \mathcal{D}_j^+ = -x^2 \partial_x + 2jx \quad \mathcal{D}_j^0 = -x \partial_x + j \quad \mathcal{D}_j^- = -\partial_x,$$

together with their complex conjugates $\bar{\mathcal{D}}_j^a$ on $\mathcal{S}(\mathbb{C})$. This representation of the zero mode algebra is then extended to a representation of the full current algebra by requiring $J_n^a P_j = 0 = \bar{J}_n^a P_j$, $n > 0$, and generating the space \mathcal{R}_j by acting with the generators J_n^a, \bar{J}_n^a , $n < 0$.

2.3. Microscopic vs. macroscopic states.

2.3.1. Macroscopic states. It is expected that a convenient plane wave basis for \mathcal{V} can be constructed in terms of wave functions $\Psi(j; x|h)$, $j \in -\frac{1}{2} + i\mathbb{R}^+$, $x \in \mathbb{C}$, that are uniquely characterized by the asymptotic behavior

$$(6) \quad \Psi\left(-\frac{1}{2} + i\rho; x|h\right) \sim e^{-\phi_0} \left(e^{-2i\rho\phi_0} \delta^2(\gamma_0 - x) + B\left(-\frac{1}{2} + i\rho\right) e^{2i\rho\phi_0} |\gamma_0 - x|^{4i\rho-2} \right) \Omega,$$

together with their descendants obtained by acting with generators J_{-n}^a, \bar{J}_{-n}^a . The function $B(j)$ that appears in (6) will be given later in (17).

The corresponding “states” $|j; x\rangle$ and their duals $\langle j; x|$ are more precisely to be understood as distributions on dense subspaces $\mathfrak{S} \subset \mathcal{V}$. From the asymptotic behavior (6) one would expect them to be delta-function normalizable, so following the terminology of [9] one may call them “macroscopic states”. It indeed turns out (cf. the Appendix A) that the normalization implied by (6) is equivalent to

$$(7) \quad \langle -\frac{1}{2} - i\rho_2; x_2 | -\frac{1}{2} - i\rho_1; x_1 \rangle = \delta^{(2)}(x_2 - x_1) \delta(\rho_2 - \rho_1).$$

2.3.2. Microscopic states. The wave-functions $\Psi(j; x|h)$ are expected to possess an analytic continuation w.r.t. the variable j ³. The “states” $|j; x\rangle, \langle j; x|$ for $j \notin -\frac{1}{2} + i\mathbb{R}$ are not even delta-function normalizable (cf. (6)) and will be called “microscopic states”. The “microscopic states” can still be understood as distributions on dense subspaces $\mathfrak{T}_j \subset \mathcal{V}$, but their domains \mathfrak{T}_j will be considerably smaller than those of the macroscopic states, e.g. not include subspaces $\mathfrak{S} \subset \mathcal{V}$ like the Schwartz-space. The difference between microscopic states and macroscopic states is crucial for their possible appearance in the spectrum: The non-appearance of microscopic states in spectral decompositions follows from rather general arguments as e.g. summarized in [11]. However, this distinction will

³See [10] for some discussion of the corresponding issue in Liouville theory

turn out (cf. Section 3.3.2 below) *not to be relevant* for the possibility of having a suitable generalization of state-operator correspondence in the H_3^+ -WZNW model.

2.3.3. Description of microscopic states in a spectral representation. In a spectral representation (4) one may characterize $\langle j, x |$ as follows. It is defined on the subspace $\mathfrak{T}_j \subset \mathcal{V}$ of vectors

$$(8) \quad |\Psi\rangle = \int_{\mathbb{C}^+} dj' \int_{\mathbb{C}} d^2x \Psi_0(j'; x) |j'; x\rangle + \text{descendants}$$

where $\Psi_0(j'; x)$ is analytic in a connected domain that contains the axis $-\frac{1}{2} + i\mathbb{R}^+$ and has the point $j' = j$ in its closure, and such that $\lim_{j' \rightarrow j} \Psi_0(j'; x) \equiv \Psi_0(j; x)$ exists. The value of the linear form $\langle j, x |$ on $|\Psi\rangle$ is then given by $\langle j, x | \Psi \rangle = \Psi_0(j; x)$.

2.3.4. Generalization. More generally one may define \mathcal{V}_{HH} to be the linear space consisting of vectors of the form

$$(9) \quad \begin{aligned} |\Psi\rangle &= |\Psi\rangle_{\text{macr}} + |\Psi\rangle_{\text{micr}} \\ &= \int_{\mathbb{C}^+} dj \int_{\mathbb{C}} d^2x \Psi_0(j; x) |j; x\rangle + \sum_{j \in \mathcal{I}} \int_{\mathbb{C}} d^2x \Psi_0(j; x) |j; x\rangle + \text{descendants}, \end{aligned}$$

where \mathcal{I} is some *finite* subset of $\mathbb{C} \setminus -\frac{1}{2} + i\mathbb{R}$. The notation \mathcal{V}_{HH} (HH for Hartle-Hawking) is used in analogy to the notation used in [9] for the space of microscopic states in Liouville theory.

An “inner product” of two microscopic states $\langle \Psi_2 |$ and $|\Psi_1\rangle$ can then be defined whenever

1. the sets \mathcal{I}_2 and \mathcal{I}_1 defined by writing $\langle \Psi_2 |$ and $|\Psi_1\rangle$ as in (9) are disjoint,
2. ${}_{\text{macr}}\langle \Psi_2 |$ is in the domain of $|\Psi_1\rangle_{\text{micr}}$ and $|\Psi_1\rangle_{\text{macr}}$ is in the domain of ${}_{\text{micr}}\langle \Psi_2 |$, and finally
3. ${}_{\text{macr}}\langle \Psi_2 | \Psi_1 \rangle_{\text{macr}}$ exists.

It is then given by

$$(10) \quad \langle \Psi_2 | \Psi_1 \rangle = {}_{\text{macr}}\langle \Psi_2 | \Psi_1 \rangle_{\text{micr}} + {}_{\text{micr}}\langle \Psi_2 | \Psi_1 \rangle_{\text{macr}} + {}_{\text{macr}}\langle \Psi_2 | \Psi_1 \rangle_{\text{macr}}$$

Although these definitions certainly call for further generalization and refinement, they will turn out to be sufficient for a consistent interpretation of microscopic intermediate states in correlation functions.

2.4. Primary fields.

2.4.1. One is interested in operators $\Phi^j(x|z)$, $x, z \in \mathbb{C}$ having the following characteristic OPE

$$(11) \quad J^a(z) \Phi^j(x|w) = \frac{1}{z-w} \mathcal{D}_j^a \Phi^j(x|w), \quad \bar{J}^a(\bar{z}) \Phi^j(x|w) = \frac{1}{\bar{z}-\bar{w}} \bar{\mathcal{D}}_j^a \Phi^j(x|w).$$

The operators $\Phi^j(x|z)$ are primary also w.r.t. the Sugawara Virasoro algebra with conformal dimensions

$$(12) \quad \Delta_j = -\frac{1}{k-2} j(j+1).$$

2.4.2. Such operators can semiclassically be identified with the following functions of the variables that appear in the H_3^+ -Lagrangian:

$$(13) \quad \Phi^j(x|z) = \frac{2j+1}{\pi} \left((\gamma - x)(\bar{\gamma} - \bar{x})e^\phi + e^{-\phi} \right)^{2j}.$$

Normal ordering will not allow the quantum operators to have such a simple form. They should however simplify for $\phi \rightarrow \infty$ where the interaction vanishes:

$$(14) \quad \Phi^j(x|z) \sim : e^{2(-j-1)\phi(z)} : \delta^2(\gamma(z) - x) + B(j) : e^{2j\phi(z)} : |\gamma(z) - x|^{4j}.$$

The operator $\delta^2(\gamma - x)$ makes sense when smeared over x since functions of $\gamma(z)$ do not require normal ordering.

2.4.3. One would expect that such operators can be constructed for any *complex* value of j . Operator-state correspondence

$$(15) \quad \lim_{z \rightarrow 0} \Phi^j(x|z)|0\rangle = |j; x\rangle \quad \lim_{z \rightarrow \infty} |z|^{4\Delta_j} \langle 0|\Phi^{-j-1}(x|z) = \langle j; x|$$

is then to be understood in the distributional sense, i.e. when evaluated against states in \mathfrak{T}_j .

2.4.4. The asymptotic expression (14) fixes a normalization for Φ^j which will be motivated in the following subsection. It will be shown in Appendix A that this normalization is equivalent to the two point function

$$(16) \quad \begin{aligned} & \langle \Phi^{-\frac{1}{2}-i\rho'}(x_2|z_2) \Phi^{-\frac{1}{2}+i\rho}(x_1|z_1) \rangle = \\ & = |z_2 - z_1|^{-4\Delta_\rho} \left(\delta^{(2)}(x_2 - x_1) \delta(\rho' - \rho) + B\left(-\frac{1}{2} + i\rho\right) |x_2 - x_1|^{2(2i\rho-1)} \delta(\rho' + \rho) \right), \end{aligned}$$

where the coefficient function $B(j)$ is explicitly given as

$$(17) \quad B(j) = -(\nu(b))^{2j+1} \frac{2j+1}{\pi} \frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))}, \quad \nu(b) = \pi \frac{\Gamma(1-b^2)}{\Gamma(1+b^2)}, \quad b^2 = \frac{1}{k-2}.$$

2.4.5. There is a linear relation between the operators $\Phi^j(x|w)$ and $\Phi^{-j-1}(x|w)$:

$$(18) \quad \Phi^j(x|z) = R(j)(\mathcal{I}_j \Phi^{-j-1})(x'|z),$$

where the *reflection amplitude* $R(j)$ is given by

$$(19) \quad R(j) = -(\nu(b))^{2j+1} \frac{\Gamma(1+b^2(2j+1))}{\Gamma(1-b^2(2j+1))},$$

and \mathcal{I}_j is the intertwining operator that establishes the equivalence of the $SL(2, \mathbb{C})$ -representations P_{-j-1} and P_j :

$$(20) \quad (\mathcal{I}_j \Phi^{-j-1})(x|z) = \frac{2j+1}{\pi} \int_{\mathbb{C}} d^2x' |x - x'|^{4j} \Phi^{-j-1}(x'|z).$$

The operator \mathcal{I}_j is normalized such that $\mathcal{I}_{-j-1} \circ \mathcal{I}_j = \text{Id}$, which implies its unitarity for $j \in -\frac{1}{2} + i\mathbb{R}$. This means that the map between in- and out-states created by the operator $\Phi^j(x|z)$ according to (15) is also unitary for these values of j .

2.5. Normalization. For applications to string theory on ADS_3 one is interested in negative values of j since $-j \equiv h$ acquires the interpretation of being the scaling dimension of an operator in the CFT living on the boundary. As one hopes this CFT to be stable, one should only have positive h in the spectrum.

The vertex operators in the H_3^+ -WZNW model can be used as ingredients for string theory vertex operators that describe scattering of strings created by sources on the boundary. For example, there is a class of vertex operators that take the form

$$(21) \quad V_n(x) = \int_{\mathbb{C}} d^2z \Phi^j(x|z) \Xi_n(z); \quad \Delta_j + \Delta_n = 1,$$

where Ξ_n is a spinless worldsheet operator in the CFT describing \mathcal{N} . Such vertex operators are interpreted as describing a string state created by a *pointlike* source located at the point x of the boundary [12], [6].

For this interpretation one needs that the leading behavior of the vertex operator $\Phi^j(x|z)$ for $\phi \rightarrow \infty$ is proportional to $\delta^2(\gamma - x)$, which is in fact found from the semiclassical expression (13) (cf. loc. cit.). It is then natural to normalize the vertex operators $\Phi^j(x|z)$ to have asymptotics (14). Semiclassically this corresponds to using the fields (13).

2.6. Three point function. The three point function of the vertex operators $\Phi^j(x|w)$ normalized by (16) takes the form

$$(22) \quad \langle \Phi^{j_3}(x_3|z_3) \Phi^{j_2}(x_2|z_2) \Phi^{j_1}(x_1|z_1) \rangle = D(j_3, j_2, j_1) C(j_3, j_2, j_1 | x_3, x_2, x_1) \cdot \\ \cdot |z_1 - z_2|^{2(\Delta_3 - \Delta_2 - \Delta_1)} |z_1 - z_3|^{2(\Delta_2 - \Delta_1 - \Delta_3)} |z_2 - z_3|^{2(\Delta_1 - \Delta_2 - \Delta_3)}.$$

2.6.1. *Structure constants* $D(j_3, j_2, j_1)$.

$$(23) \quad D(j_3, j_2, j_1) = \frac{G(j_1 + j_2 + j_3 + 1) G(j_1 + j_2 - j_3) G(j_1 + j_3 - j_2) G(j_2 + j_3 - j_1)}{(\nu(b))^{-j_1 - j_2 - j_3 - 1} G_0 G(2j_1 + 1) G(2j_2 + 1) G(2j_3 + 1)}.$$

The special function $G(j)$ is related to the Υ -function introduced in [13] via $G(j) = b^{-b^2x(x+1+b^{-2})} \Upsilon^{-1}(-bj)$ and

$$(24) \quad G_0 = -2\pi^2 \frac{\Gamma(1+b^2)}{\Gamma(-b^2)} G(-1)$$

The Υ -function was in loc. cit. defined by an integral representation which converges in the strip $0 < \text{Re}(x) < Q/2$ and defines an analytic function there. It may alternatively be constructed out of the Barnes Double Gamma function [14][15] as follows:

$$\Upsilon^{-1}(s) = \Gamma_2(s|b, b^{-1}) \Gamma_2(b + b^{-1} - s|b, b^{-1}), \\ \log \Gamma_2(s|\omega_1, \omega_2) = \lim_{t \rightarrow 0} \frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} (s + n_1\omega_1 + n_2\omega_2)^{-t}.$$

2.6.2. *The coefficients* $C(j_3, j_2, j_1 | x_3, x_2, x_1)$.

$$(25) \quad C(j_3, j_2, j_1 | x_3, x_2, x_1) = |x_1 - x_2|^{2(j_1 + j_2 - j_3)} |x_1 - x_3|^{2(j_1 + j_3 - j_2)} |x_2 - x_3|^{2(j_2 + j_3 - j_1)}$$

They are Clebsch-Gordan coefficients⁴ for the decomposition of the tensor product $P_{j_2} \otimes P_{j_1}$ of $SL(2, \mathbb{C})$ principal series representations, cf. [16].

⁴More precisely: distributional kernel

The x_i -dependence of the $C(j_3, j_2, j_1|x_3, x_2, x_1)$ is uniquely fixed by $SL(2, \mathbb{C})$ -invariance as long as none of $j_1 + j_2 - j_3$, $j_1 + j_3 - j_2$, $j_2 + j_3 - j_1$ equals $-n - 1$, $n = 0, 1, 2, \dots$

2.7. Operator product expansion. The following result [4] provides the basis for a complete determination of the operator product expansion for any product $\Phi^{j_2}\Phi^{j_1}$, $\Phi^{j_i} = \Phi^{j_i}(x_i, \bar{x}_i|z_i, \bar{z}_i)$, $i = 1, 2$:

There is a range \mathcal{R} for the values of j_2, j_1 given by the inequalities

$$(26) \quad |\operatorname{Re}(j_{21}^\pm)| < \frac{1}{2} \quad j_{21}^+ = j_2 + j_1 + 1, \quad j_{21}^- = j_2 - j_1.$$

such that the operator product expansion takes the form

$$(27) \quad \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) = \int_{\mathcal{C}^+} dj_3 \ D_{21}(j_3) |z_2 - z_1|^{\Delta_{21}(j_3)} (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) \\ + \text{descendants},$$

where the operator $(\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1)$ has been defined as

$$(28) \quad (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) \equiv \int_{\mathbb{C}} d^2x_3 \ C(j_3, j_2, j_1|x_3, x_2, x_1)\Phi^{-j_3-1}(x_3|z_1)$$

and the abbreviations $D_{21}(j_3) = D(j_3, j_2, j_1)$ and $\Delta_{21}(j_3) = \Delta_{j_3} - \Delta_{j_2} - \Delta_{j_1}$ have been introduced for later convenience. A more precise description of the descendant contributions is given in Appendix B.

The region (26) will turn out to be precisely the maximal region in which j_1, j_2 may vary such that none of the poles of the integrand in (27) hits the contour \mathcal{C} of integration over j_3 in the OPE.

3. ANALYTIC CONTINUATION OF THE OPE

The aim of the present section is to demonstrate that the operator product expansion (27) admits an analytic continuation to generic complex values of j_1, j_2 .

3.1. Analytic properties of the integrand in (27).

3.1.1. Reflection property. It will be important to note that the integrand of the j_3 -integration in (27) is symmetric under $j_3 \rightarrow -j_3 - 1$. This can be seen as a consequence of the reflection relation (18) when used to express the operator Φ^{-j_3-1} in terms of Φ^{j_3} . The integral over x_3 in (28) may then be carried out by using the integral (see e.g. [17])

$$(29) \quad \int_{\mathbb{C}} d^2t \ |t|^{2a} |1-t|^{2b} = -\pi \frac{\gamma(-1-a-b)}{\gamma(-a)\gamma(-b)}, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}.$$

The resulting functional equation for $(\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1)$ can be written as

$$(30) \quad (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) = \\ = -\frac{1}{\pi} \frac{\gamma(-2j_3)}{\gamma(j_1-j_2-j_3)\gamma(j_2-j_1-j_3)} \frac{1}{B(j_3)} (\mathcal{J}_{21}(-j_3-1)\Phi^{j_3})(z_1).$$

The structure constants on the other hand satisfy the identity

$$(31) \quad D_{21}(j_3) = -\pi \frac{\gamma(j_1-j_2-j_3)\gamma(j_2-j_1-j_3)}{\gamma(-2j_3)} B(j_3) D_{21}(-j_3-1),$$

which can be easily verified using the functional equations $G(x) = G(-x - 1 - b^{-2})$ and

$$(32) \quad G(j-1) = \frac{\Gamma(1+b^2j)}{\Gamma(-b^2j)}G(j), \quad G(j-b^{-2}) = b^{2(2j+1)}\frac{\Gamma(1+j)}{\Gamma(-j)}G(j),$$

which follow from the corresponding functional equations of the Υ -function given in [13]. The relations (30) and (32) imply the symmetry of the integrand in (27) under $j_3 \rightarrow -j_3 - 1$.

3.1.2. Poles of the structure constants. To obtain the meromorphic continuation of the structure constants $D(j_3, j_2, j_1)$ one may note that the function $G(j)$ admits a meromorphic continuation to the complex j plane which may be defined by the functional relations (32). It follows that the function $G(j)$ has poles for $j = n + mb^{-2}$ and $j = -(n+1) - (m+1)b^{-2}$, $n, m = 0, 1, 2, \dots$. The resulting set of poles of the structure constants is given by

$$(33) \quad \begin{aligned} j_3 &= j_{21}^+ - 1 - n - mb^{-2} & j_3 &= j_{21}^+ + n + (m+1)b^{-2} \\ j_3 &= j_{21}^- - (n+1) - (m+1)b^{-2} & j_3 &= j_{21}^- + n + mb^{-2} & n, m &= 0, 1, 2, \dots \\ j_3 &= -j_{21}^\pm - (n+1) - (m+1)b^{-2} & j_3 &= -j_{21}^\pm + n + mb^{-2} \end{aligned}$$

3.1.3. Poles of the operator $\mathcal{J}_{21}(j_3)\Phi^{-j_3-1}$. Concerning the meromorphic continuation of the operator $(\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1)$ one should first note that the distribution $|x|^{2j}$ is meromorphic in j with poles at $j = -n - 1$, $n = 0, 1, 2, \dots$ and residues

$$(34) \quad \text{Res}_{j=-n-1} |x|^{2j} = \frac{\pi}{(n!)^2} \delta^{(n,n)}(x) \equiv \frac{\pi}{(n!)^2} (\partial_x \partial_{\bar{x}})^n \delta(x),$$

see e.g. [22]. One consequently finds poles of the OPE coefficients for $j_3 = j_{21}^+ + n$, $j_3 = \pm j_{21}^- - n - 1$.

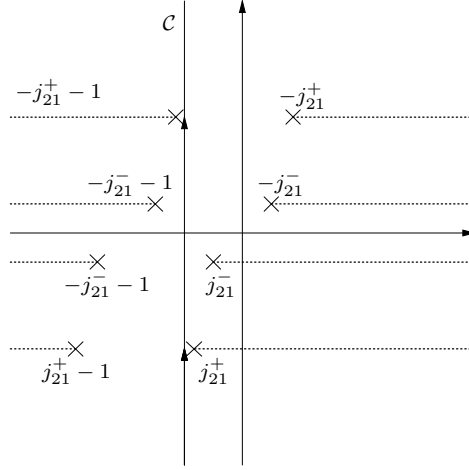
A further series of poles arises from the integration over x_3 in (28). These are related to the poles from the distributions $|x|^{2j}$ by the reflection $j_3 \rightarrow -j_3 - 1$.

3.1.4. It is important to note that the descendant contributions to the OPE do not introduce further poles with positions depending on j_1, j_2 , as follows from the results in Appendix B.

Alltogether one finds the following set $\mathfrak{P}(j_2, j_1)$ of poles ($n, m = 0, 1, 2, \dots$)

$$(35) \quad \begin{aligned} j_3 &= j_{21}^\pm - 1 - n - mb^{-2} & j_3 &= j_{21}^\pm + n + mb^{-2} \\ j_3 &= -j_{21}^\pm - 1 - n - mb^{-2} & j_3 &= -j_{21}^\pm + n + mb^{-2} \end{aligned}$$

It is useful to visualize the position of the poles in the complex j_3 -plane:



The crosses symbolize the first poles to the left (right) of the contour \mathcal{C} , the dashed lines indicate the lines along which further poles follow with spacing 1 and b^{-2} .

3.2. Continuing the integral over j_3 in (27). When attempting to continue beyond the region given by (26) one finds poles that hit the contour of integration \mathcal{C}^+ . However, as long as the imaginary parts of j_{21}^+ and j_{21}^- do not vanish one may define the analytic continuation of (27) by deforming the contour \mathcal{C}^+ . The deformed contour may always be represented as the sum of the original contour and finitely many circles around the poles. These latter lead to a finite sum of residue contributions to the OPE. The important case where j_{21}^\pm are real can be treated by giving them a small imaginary part which is sent to zero after having deformed the contour. The result does not depend on the sign of the imaginary part as a consequence of the symmetry of the integrand in (27) under $j_3 \rightarrow -j_3 - 1$.

Only the case $j_{21}^+ < -\frac{1}{2}$ will be described explicitly here since the generalization to the other cases is straightforward. For this case one finds an expression of the following form:

$$\begin{aligned}
 \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) &= \sum_{j_3 \in \mathcal{P}_{21}^+} z_{21}^{\Delta_{21}(j_3)} D_{21}(j_3) (\mathcal{J}_{21}^R(j_3)\Phi^{-j_3-1})(z_1) \\
 &+ \sum_{j_3 \in \mathcal{P}_{21}^-} z_{21}^{\Delta_{21}(j_3)} D_{21}^R(j_3) (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) \\
 &+ \sum_{\varepsilon \in \{+, -\}} \sum_{j_3 \in \mathcal{Q}_{21}^\pm} z_{21}^{\Delta_{21}(j_3)} D_{21}^R(j_3) (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) \\
 &+ \int_{\mathcal{C}^+} dj_3 z_{21}^{\Delta_{21}(j_3)} D_{21}(j_3) (\mathcal{J}_{21}(j_3)\Phi^{-j_3-1})(z_1) \\
 &+ \text{descendants},
 \end{aligned}
 \tag{36}$$

where the ranges of summations are given by the sets

$$\begin{aligned}
 \mathcal{P}_{21}^\pm &= \{j_3 = j_{21}^\pm + n \mid 0 \leq n; \operatorname{Re}(j_3) < -\tfrac{1}{2}\}, \\
 \mathcal{Q}_{21}^\pm &= \{j_3 = j_{21}^\pm + n + (m+1)b^{-2} \mid 0 \leq n, m; \operatorname{Re}(j_3) < -\tfrac{1}{2}\},
 \end{aligned}
 \tag{37}$$

and $D_{21}^R(j)$, $\mathcal{J}_{21}^R(j)$ denote the residues of $D_{21}(j)$, $\mathcal{J}_{21}(j)$ when j is a pole of these functions.

It is interesting to note that the residue $\mathcal{J}_{21}^R(j_{21}^+ + n)$ contains delta-functions as a consequence of (34):

$$(38) \quad \mathcal{J}_{21}^R(j_{21}^+ + n) = \frac{\pi}{(n!)^2} \delta(x_{21}) |x_{31}|^{2(2j_1+n+1)} |x_{32}|^{2(2j_2+n+1)}.$$

Finally, one may observe that all terms that appear in the OPE are meromorphic in j_2 , j_1 with set of poles $\mathfrak{S}(j_2, j_1)$ given by

$$(39) \quad 2j_{21}^\pm = -(n+1) - (m+1)b^{-2}.$$

3.3. General remarks.

3.3.1. The leading term for $z_2 \rightarrow z_1$ appears in the first term of (36). It takes the following simple form:

$$(40) \quad \begin{aligned} \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) &= |z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \cdot \\ &\cdot \left(\delta^2(x_2 - x_1)\Phi^{j_1+j_2+1}(x_1|z_1) + \text{terms vanishing for } z_2 \rightarrow z_1 \right). \end{aligned}$$

3.3.2. It is worth noting that the state $\Psi_{21} \equiv \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1)|0\rangle$ will be in the domain of the microscopic state $\langle -j_3 - 1|$ as defined in Section 2.3.3 as long as one does not happen to hit one of the poles (35). One then has

$$(41) \quad \langle -j_3 - 1|\Psi_{21}\rangle = \lim_{z_3 \rightarrow \infty} |z|^{4\Delta_j} \langle 0|\Phi^{j_3}(x_3|z_3)\Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1)|0\rangle,$$

which illustrates how the remarkable analyticity properties of the three point function make the distinction between macroscopic and microscopic states irrelevant for the issue of state-operator correspondence.

3.3.3. The OPE gets singular for the values of j_2 , j_1 given in (39). This is due to the fact that the “outgoing” representation becomes degenerate for these values. In these cases there do not exist primary fields unless j_2 , j_1 are further restricted to satisfy the corresponding fusion rules.

3.4. Interesting special cases.

3.4.1. For the special case $j_2 = -1$ one thereby obtains the OPE of the operator that has appeared in [6] as the basic building block for the construction of the space-time current- and Virasoro algebras. The important point is that there does not appear a factor which depends on j_1 , as was conjectured in [6]. This is hereby verified on the basis of the structure constants determined in [4].

3.4.2. It is interesting to observe that the OPE simplifies in the special case that j_2 and j_1 satisfy the so-called unitarity bound [1][18]

$$(42) \quad -1 - \frac{1}{2b^2} < j_i < 0, \quad i = 1, 2$$

and $k > 3$. It only involves residue terms corresponding to poles in \mathcal{P}_{21}^\pm . Up to terms containing $\delta^2(x_2 - x_1)$ and derivatives thereof (“contact terms”) one only finds terms containing operators Φ^{-j_3-1} , where j_3 also satisfies the unitarity bound.

3.4.3. Another interesting case is that of integer level k . The poles at $j_3 \in \mathcal{Q}_{21}^\pm$, $m > 0$ degenerate into double poles for this case, so that they do not lead to contributions in the OPE.

4. FACTORIZATION OF THE FOUR POINT FUNCTION

4.1. Case with only macroscopic intermediate states. One may start with the case that

$$(43) \quad \begin{aligned} |\operatorname{Re}(j_{21}^\pm)| &< \frac{1}{2} & j_{21}^+ &= j_2 + j_1 + 1, & j_{21}^- &= j_2 - j_1 \\ |\operatorname{Re}(j_{43}^\pm)| &< \frac{1}{2} & j_{43}^+ &= j_4 + j_3 + 1, & j_{43}^- &= j_4 - j_3. \end{aligned}$$

In this case one may obtain the factorization e.g. in the s-channel one may use the OPE (27) for the pairs of operators $\Phi^{j_4}\Phi^{j_3}$ and $\Phi^{j_2}\Phi^{j_1}$ and observe that only macroscopic intermediate states are produced as a consequence of (43).

The resulting decomposition of the four point function can be written as (see Appendix B for some details)

$$(44) \quad \langle \Phi^{j_4} \dots \Phi^{j_1} \rangle = \int_{\mathbb{C}^+} dj \, D_{43}(j) B(-j-1) D_{21}(j) \mathcal{G}_j(J|X|Z).$$

The function $\mathcal{G}_j(J|X|Z)$, $J = (j_4, \dots, j_1)$, $X = (x_4, \dots, x_1)$, $Z = (z_4, \dots, z_1)$ that appears in (44) will be called *non-chiral block*. It takes the following form:

$$(45) \quad \begin{aligned} \mathcal{G}_j(J|X|Z) &= |z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)} |z_{42}|^{-4\Delta_2} |z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)} \cdot \\ &\cdot |z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)} |z|^{2\Delta_{21}(j)} \mathcal{O}_j(J|X|z) \overline{\mathcal{O}_j(J|X|z)} G_j(J|X), \end{aligned}$$

where the following objects have been introduced: There is first of all the usual cross-ratio $z = \frac{z_{21}z_{43}}{z_{31}z_{42}}$.

The function $G_j(J|X)$ represents the summation over the zero mode subspace P_j of the representation \mathcal{R}_j and is given by the integral

$$(46) \quad G_j(J|X) = \int_{\mathbb{C}} d^2x d^2x' \frac{C(j_4, j_3, j|x_4, x_3, x) C(j, j_2, j_1|x', x_2, x_1)}{|x - x'|^{4j+4}}.$$

The operators $\mathcal{O}_j(J|X|z)$ finally are given as a formal power series in z

$$(47) \quad \mathcal{O}_j(J|X|z) = \sum_{n=0}^{\infty} z^n \mathcal{D}_j^{(n)}(J|X),$$

where the $\mathcal{D}_{x,j}^{(n)}(J|x)$ are differential operators containing derivatives w.r.t. x_4, \dots, x_1 of finite order. These operators are not known explicitly, but the construction given in Appendix B will allow to obtain some important information.

4.2. Chiral factorization.

4.2.1. The integral (46) can be simplified by exploiting $SL(2, \mathbb{C})$ -invariance

$$(48) \quad G_j(J|X) = |x_{43}|^{2(j_4+j_3-j_2-j_1)} |x_{42}|^{4j_2} |x_{41}|^{2(j_4+j_1-j_2-j_3)} |x_{31}|^{2(j_3+j_2+j_1-j_4)} G_j(J|x),$$

where $x = \frac{x_{21}x_{43}}{x_{31}x_{42}}$, and the integral for $G_j(J|x)$ obtained by putting $x_4 = \infty$, $x_3 = 1$, $x_2 = x$, $x_1 = 0$ can be carried out by using the integrals calculated in [17], pp. 152:

$$(49) \quad G_j(J|x) = \frac{\pi^2}{(2j+1)^2} |F_j(J|x)|^2 - \frac{\pi^2}{\gamma(2j+2)} \frac{\gamma(1+j+j_4-j_3)\gamma(1+j+j_3-j_4)}{\gamma(j_1-j_2-j)\gamma(j_2-j_1-j)} |F_{-j-1}(J|x)|^2,$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ and

$$(50) \quad F_j(J|x) \equiv x^{j_1+j_2-j} F(j_1-j_2-j, j_4-j_3-j; -2j; x).$$

4.2.2. The γ -functions in front of the second term of (49) can be absorbed by using (31). It follows that the terms in (44) that originate from the second term in (49) are related to the terms from the first one by $j \rightarrow -j-1$. This allows to extend the integration over the half-axis \mathcal{C}^+ in (44) to an integration over the full axis $\mathcal{C} = -\frac{1}{2} + i\mathbb{R}$. The resulting expression shows a *holomorphically factorized* form

$$(51) \quad \langle \Phi^{j_4} \dots \Phi^{j_1} \rangle = \int_{\mathcal{C}} dj \ D_{43}(j) \frac{1}{B(j)} D_{21}(j) |\mathcal{F}_j(J|X|Z)|^2,$$

where the *chiral blocks* $\mathcal{F}_{js}(J|X|Z)$ are of the form

$$(52) \quad \begin{aligned} \mathcal{F}_j(J|X|Z) &= z_{43}^{\Delta_2+\Delta_1-\Delta_4-\Delta_3} z_{42}^{-2\Delta_2} z_{41}^{\Delta_3+\Delta_2-\Delta_4-\Delta_1} z_{31}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3} \cdot \\ &\cdot x_{43}^{j_4+j_3-j_2-j_1} x_{42}^{2j_2} x_{41}^{j_4+j_1-j_2-j_3} x_{31}^{j_1+j_2+j_3-j_4} F_j(J|x|z) \\ F_j(J|x|z) &= z^{\Delta_{21}(j)} \mathcal{O}_j(J|x|z) F_j(J|x). \end{aligned}$$

4.3. Properties of conformal blocks.

4.3.1. *KZ-equations.* The 4-point functions satisfy a system of partial differential equations that is formally similar to the Knizhnik-Zamolodchikov equations for the $SU(2)$ -WZNW model (cf. Appendix C):

$$(53) \quad \partial_{z_r} F_j(J|X|Z) = \frac{1}{k-2} \sum_{r < s} \frac{\mathcal{D}_{rs}}{z_r - z_s} F_j(J|X|Z),$$

where $r, s = 1, \dots, 4$ and the differential operators \mathcal{D}_{rs} are given by

$$(54) \quad \mathcal{D}_{rs} = -\mathcal{D}_{jr}^0 \mathcal{D}_{js}^0 + \frac{1}{2} (\mathcal{D}_{jr}^+ \mathcal{D}_{js}^- + \mathcal{D}_{jr}^- \mathcal{D}_{js}^+)$$

In the case of the four point function one may reduce it to an equation for the cross-ratios x and z which was first considered in the context of the $SU(2)$ WZNW model in [19]:

$$(55) \quad \partial_z F_j(J|X|Z) = \frac{1}{k-2} \left(\frac{P}{z} + \frac{Q}{z-1} \right) F_j(J|X|Z),$$

where $(\kappa \equiv j_1 + j_2 + j_3 - j_4)$

$$(56) \quad \begin{aligned} P &= x^2(x-1)\partial_x^2 - ((\kappa-1)x^2 - 2j_1x + 2j_2x(x-1))\partial_x + 2\kappa j_2x - 2j_1j_2 \\ Q &= -(1-x)^2x\partial_x^2 + ((\kappa-1)(1-x)^2 - 2j_3(1-x) + 2j_2x(x-1))\partial_x \\ &\quad + 2\kappa j_2(1-x) - 2j_3j_2 \end{aligned}$$

4.3.2. *X- and Z-dependence of the chiral blocks.* The chiral blocks are expected to be multivalued analytic functions of the variables z_4, \dots, z_1 and x_4, \dots, x_1 on $\mathbb{C}^4 \setminus \{z_r = z_s\} \times \mathbb{C}^4 \setminus \{x_r = x_s\}$. This will follow from the KZ-equations (53) once the convergence of the formal power series that represent the conformal blocks (cf. Appendix C) is established.

Projective invariance allows to easily obtain the singularity structure from the singularities of $F_j(J|x|z)$. These coincide with the singular points of (55), which are at $z = 0, 1, x, \infty$ and $x = 0, 1, z, \infty$. The singular behavior near $z = 0, 1, \infty$ is of the usual form as e.g. expressed by the last line in (52).

Interesting is also the singular behavior near $x = 0, 1, z, \infty$. This can also be determined from (55). One finds e.g.

$$(57) \quad \begin{aligned} F_j(J|X|Z) &\sim \mathcal{O}(1) + \mathcal{O}(x_{21}^{j_1+j_2-j_3-j_4}) \quad \text{near } x = 0, \\ F_j(J|X|Z) &\sim \mathcal{O}(1) + \mathcal{O}(x_{21}^{j_1+j_2+j_3+j_4+k}) \quad \text{near } x = z. \end{aligned}$$

The exponents at the singular points $x = 0, 1, z, \infty$ do not depend on j .

4.3.3. *J- and j-dependence of the non-chiral blocks.* An important consequence of the representation of the chiral/non-chiral blocks in the form (45)(52) is that it allows to control the J - and j -dependence of these objects. It follows from the construction (cf. Appendix B) of the differential operators that appear in the expansion (47) of the operator $\mathcal{O}_j(J|X|z)$ that $\mathcal{D}_{x,j}^{(n)}(J|x)$ depends *polynomially* on j_4, \dots, j_1 and *rationally* on j . $\mathcal{D}_{x,j}^{(n)}(J|x)$ has only simple poles at the values $j = j_{r,s}^\pm$, $2j_{r,s}^\pm + 1 = \pm(r + b^{-2}s)$. Provided that the formal power series representation for $F_j(J|x|z)$ implied by (52),(47) actually converges, one concludes that up to the poles at $j = j_{r,s}^\pm$ introduced by the descendant contributions one may read off the analyticity properties w.r.t. J and j from the lowest order terms $G_j(J|X)$ or $F_j(J|X)$. In the case of $G_j(J|X)$ one finds from the expressions (48),(49) that the set of poles in the j -dependence with positions depending on J is just the union of the sets of poles of $C(j_4, j_3, j|x_4, x_3, x)$ and $C(j, j_2, j_1|x', x_2, x_1)$, which were discussed in Section 3.1.3.

4.4. Comments.

4.4.1. The operators $\mathcal{O}_j(J|X|z)$ are entirely determined by the structure of the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}})$. In particular, they do not depend on the choice of representation for the zero modes.

These operators may be seen as affecting the *deformation* of the semiclassical chiral block $F_j(J|X)$ (cf. [8]) into the full conformal block $F_j(J|X|z)$.

Equivalently one may view it in connection to the KZ-equations as a kind of *generalized wave operator* that maps solutions $z^{\Delta_{21}(j)} F_j(J|x)$ of the “free wave equation” $(k-2)z\partial_z F^{(0)} = P F^{(0)}$ into solutions $F_j(J|X|z)$ of (55).

4.4.2. In the case of the H_3^+ -WZNW-model it is not a priori clear that the four point function possesses a holomorphic factorization of the form (51) since the representations of the current algebra that appear in the spectral decomposition (4) do *not* show a holomorphic splitting as tensor product $\mathcal{V}_L^j \otimes \mathcal{V}_R^j$, where \mathcal{V}_L^j (\mathcal{V}_R^j) are irreducible representations of the current algebras generated by the J_n^a (\bar{J}_n^a) respectively. It is the representation of the

zero modes J_0^a, \bar{J}_0^a which does not factorize as tensor product of separate representations for J_0^a and \bar{J}_0^a respectively.

This is related to the fact that the integrand in (51) for any single value does not contain all contributions of an intermediate representation \mathcal{R}_j . It is only the sum of the integrand taken at j and $-j - 1$ which represents the sum over contributions from \mathcal{R}_j .

4.4.3. The KZ-equations for the four point functions can not be reduced to a system of ordinary differential equations as in the case of the $SU(2)$ -WZNW model studied in [19].

The problem of constructing a complete (in an appropriate sense) set of solutions to the KZ-equations that can be identified with the chiral blocks is therefore rather nontrivial. This seems to be the main open problem towards making the bootstrap approach to the H_3^+ -WZNW model mathematically rigorous. What can be shown at present (cf. Appendix C) is that there exist unique solutions in the sense of formal power series in z which can be identified with chiral blocks.

4.4.4. The information on the singularities in the x -dependence of four point functions is relevant for string theory on ADS_3 for the following reason: If one tries to determine the OPE in the boundary CFT that is supposed to describe the space-time physics of string theory on ADS_3 , one might consider e.g. the behavior of $V_{n_2}(x_2)V_{n_1}(x_1)$ for $x_2 \rightarrow x_1$. This should be encoded in the four point function

$$\mathcal{F}(N|X) \equiv \langle V_{n_2}(x_4)V_{n_1}(x_3)V_{n_2}(x_2)V_{n_1}(x_1) \rangle,$$

where $N = (n_2, n_1, n_2, n_1)$. The genus zero contribution to this four point function can be represented as

$$(58) \quad \mathcal{F}(N|X) = \int d^2z \langle \Phi^{j_2}(x_4|\infty)\Phi^{j_1}(x_3|1)\Phi^{j_2}(x_2|z)\Phi^{j_1}(x_1|0) \rangle_{H_3^+} \cdot \\ \cdot \langle \Xi_{n_2}(\infty)\Xi_{n_1}(1)\Xi_{n_2}(z)\Xi_{n_1}(0) \rangle_{\mathcal{N}},$$

where the $j_i, i = 1, 2$ are determined by the mass-shell condition in (21), and $\langle \dots \rangle_{\mathcal{N}}$ represents a correlation function in the CFT describing \mathcal{N} .

Equation (57) now confirms the expectation [6] that singularities of $\mathcal{F}(N|X)$ for $x_2 \rightarrow x_1$ occur *only* from the part of the region of integration in (58) where $z \rightarrow 0$, which corresponds to $\Phi^{j_2}(x_2|z_2)\Xi_{n_2}(z_2)$ approaching $\Phi^{j_1}(x_1|z_1)\Xi_{n_1}(z_1)$ on the *worldsheet*.

4.5. Cases with microscopic intermediate states.

4.5.1. If j_4, \dots, j_1 are no longer restricted by (43) then at least one of $\langle \Psi_{43} | \equiv \langle 0 | \Phi^{j_4} \Phi^{j_3}$ and $|\Psi_{21} \rangle \equiv \Phi^{j_2} \Phi^{j_1} | 0 \rangle$ will be microscopic.

The decomposition of the four point function as sum over non-chiral blocks in the case of generic j_4, \dots, j_1 can be obtained by using the results of Section 3 for the OPE's $\Phi^{j_4} \Phi^{j_3}$ and $\Phi^{j_2} \Phi^{j_1}$ in the generic case. It follows from the remarks in Section 4.3.3 that the result will be just the same as obtained by directly constructing the analytic continuation of the expression (44) by the techniques of Section 3.

The resulting residue contributions are well-defined and non-singular as long as the sets $\mathcal{S}_{21} \equiv \mathcal{P}_{21}^+ \cup \mathcal{P}_{21}^- \cup \mathcal{Q}_{21}^+ \cup \mathcal{Q}_{21}^-$ and \mathcal{S}_{43} are disjoint. This condition is equivalent to conditions (1) and (2) of Section 2.3.4 for the existence of the inner product $\langle \Psi_{43} | \Psi_{21} \rangle$.

This shows that intermediate contributions of microscopic intermediate states are indeed perfectly well-defined in the generalization of the bootstrap framework that was used in the present paper.

4.5.2. The four point function develops poles for the set of j_4, \dots, j_1 such that $\mathcal{S}_{21} \cap \mathcal{S}_{43} \neq \emptyset$. These cases are analogous to the “resonant” amplitudes in Liouville theory (cf. e.g. [20]). Representations like (44) or (51) for the residues at these poles can of course easily be worked out from the results of the present paper. The integration over j disappears in these cases. It should be possible to represent these residues more explicitly by free field methods.

5. APPENDIX A: COMPARISON OF NORMALIZATIONS

The aim of this appendix is to show that the normalization underlying the expression (23) for the structure constants indeed coincides with the one implied by the $\phi \rightarrow \infty$ -asymptotics (14). This is furthermore compared to the normalization that was used in [4].

5.1. Comparison via leading order OPE.

5.1.1. The leading order OPE can on the one hand be directly obtained from the $\phi \rightarrow \infty$ -asymptotics (14): For the product of two operators $\Phi^{j_2}(x_2|z_2)$ and $\Phi^{j_1}(x_1|z_1)$ one finds that the $\phi \rightarrow \infty$ -asymptotics is given by

$$(59) \quad \begin{aligned} & \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) \sim \\ & \sim \delta^2(x_2 - x_1)|z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \left[\delta^2(\gamma - x_1) : e^{2(-j_1-j_2-2)\Phi(z)} : \right], \end{aligned}$$

where it was assumed that $j_i < -\frac{1}{2}$. The state created by the operator on the right hand side will be non-normalizable provided that $j_1 + j_2 + 1 < -\frac{1}{2}$. In that case there is a unique microscopic operator that has asymptotic behavior equal to the term in square brackets that appears on the right hand side of (59):

$$(60) \quad \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) \sim \delta^2(x_2 - x_1)|z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \Phi^{j_1+j_2+1}(x_1|z_1).$$

Terms that are subleading for $\phi \rightarrow \infty$ lead to contributions in the OPE that are subleading for $z_2 \rightarrow z_1$.

5.1.2. This should be compared with the leading order asymptotics that follows from the structure constants (23) by using the methods of Sect. 3. If one would keep G_0 and $\nu(b)$ in (23) arbitrary one would find instead of (40) the expression

$$(61) \quad \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) \sim 2\pi^3 \frac{G(-1)b^2}{G_0\nu(b)} \delta^2(x_2 - x_1)|z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \Phi^{j_1+j_2+1}(x_1|z_1).$$

This means that comparison of leading order asymptotics can only fix the product $G_0\nu(b)$.

5.2. **Comparison via two-point function.** The two-point function of the operators $\Theta^j(x|z)$ is recovered as the limit

$$(62) \quad \lim_{\epsilon \rightarrow 0} \langle \Phi^{-\frac{1}{2}-i\rho'}(x_2|z_2) \Phi^\epsilon(x|z) \Phi^{-\frac{1}{2}+i\rho}(x_1|z_1) \rangle,$$

which requires a little care due to the distributional nature of these objects. The limit of course vanishes for $\rho \neq \rho'$ due to the factor $G^{-1}(2\epsilon + 1)$. It will be assumed that $\rho, \rho' > 0$ in the following. There are however two further terms that behave singular for $\rho - \rho' \simeq \epsilon \rightarrow 0$: The factors $|x_3 - x_1|^{2(i(\rho-\rho')-1-\epsilon)}$ and $G(\epsilon + i(\rho - \rho'))$. The singular behavior of the former can be represented by $(i(\rho - \rho') - i\epsilon)^{-1} \pi \delta^{(2)}(x_2 - x_1)$, (cf. (34)), whereas the latter behaves as $(\epsilon + i(\rho - \rho'))^{-1} \text{Res}_{x=0} G(x)$. The limit (62) is therefore given by

$$(63) \quad |z_2 - z_1|^{-4\Delta_\rho} \delta^{(2)}(x_2 - x_1) \frac{G(-1)}{G_0} \frac{\Gamma(1+b^2)}{\Gamma(-b^2)} \lim_{\epsilon \rightarrow 0} \frac{2\epsilon}{\epsilon^2 + (\rho - \rho')^2}.$$

The last factor yields $2\pi\delta(\rho - \rho')$, so that one would find the two point function to be

$$(64) \quad \langle \Phi^{-\frac{1}{2}-i\rho'}(x_2|z_2) \Phi^{-\frac{1}{2}+i\rho}(x_1|z_1) \rangle = -2\pi^2 \frac{G(-1)}{G_0} \frac{\Gamma(1+b^2)}{\Gamma(-b^2)} \delta(\rho - \rho') \delta^{(2)}(x_2 - x_1)$$

Requiring the prefactors in (61) and (64) to be unity uniquely determines G_0 and $\nu(b)$ to be as given in Sect. 2.

5.3. Normalization in [4]. The normalization that was underlying the final result for the three point function [4] may be seen to be more natural if one is interested in values of $j > -\frac{1}{2}$: In that case it is the second term in (14) that dominates the $\phi \rightarrow \infty$ -asymptotics, so it is natural to require this term to have prefactor one. The operators defined by this normalization will be denoted $\Theta^j(x|z)$.

The leading order OPE may then, as in the previous subsection, be determined from the $\phi \rightarrow \infty$ -asymptotics:

$$(65) \quad \begin{aligned} & \Theta^{j_2}(x_2|z_2) \Theta^{j_1}(x_1|z_1) \sim \\ & \sim |z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} |\gamma - x_2|^{4j_2} |\gamma - x_1|^{4j_1} : e^{2(j_1+j_2)\Phi(z)} : \\ & \sim |z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \left[|\gamma - x_1|^{4(j_1+j_2)} : e^{2(j_1+j_2)\Phi(z)} : \right] + \mathcal{O}(x_2 - x_1) \\ & \sim |z_2 - z_1|^{-4b^2(j_1+1)(j_2+1)} \Theta^{j_1+j_2}(x_1|z_1) + \mathcal{O}(x_2 - x_1) \end{aligned}$$

This is to be compared to the result that follows from the structure constants of the operators $\Theta^j(x|z)$ by applying the method of Sect.3. These structure constants were given in formula (64) of [4]. In terms of the functions G used in the present paper one may rewrite the result as

$$(66) \quad \tilde{D}(j_3, j_2, j_1) = \frac{G(j_1 + j_2 + j_3 + 1) G(j_1 + j_2 - j_3) G(j_1 + j_3 - j_2) G(j_2 + j_3 - j_1)}{(\nu'(b))^{-j_1-j_2-j_3-1} G'_0 G(2j_1) G(2j_2) G(2j_3)}.$$

The freedom that was left over by the analysis in [4] is parametrized by $\nu'(b)$ and G'_0 , which had not been determined there.

Before using the method of Section 3. to determine the leading term in the OPE one needs to determine the precise form of the OPE in the case that j_1, j_2 are in the range (26) where only macroscopic operators appear. To this aim one needs the two-point function following from (66). This is found to be

$$(67) \quad \langle \Theta^{-\frac{1}{2}-i\rho'}(x_2|z_2) \Theta^{-\frac{1}{2}+i\rho}(x_1|z_1) \rangle = \left[-2\pi^2 b^{-4} \frac{G(-1)}{G'_0} \right] \frac{1}{4\rho^2} \delta(\rho - \rho') \delta^{(2)}(x_2 - x_1).$$

Requiring the prefactor in square brackets to be unity determines G'_0 . It is important to note the factor $\frac{1}{4\rho^2}$ in (67): Due to this factor one finds for the OPE of operators $\Theta^{j_2}(x_2|z_2) \Theta^{j_1}(x_1|z_1)$ in the case that j_2, j_1 are in (26) a form that differs from (27):

$$(68) \quad \begin{aligned} \Theta^{j_2}(x_2|z_2) \Theta^{j_1}(x_1|z_1) = & -\frac{1}{2} \int_{\mathbb{C}} dj_3 (2j_3 + 1)^2 |z_2 - z_1|^{\Delta_3 - \Delta_2 - \Delta_1} \tilde{D}(j_3, j_2, j_1) \cdot \\ & \cdot \int_{\mathbb{C}} d^2 x_3 C(j_3, j_2, j_1 | x_3, x_2, x_1) \Theta^{-j_3-1}(x_3|z_1) \\ & + \text{descendants}, \end{aligned}$$

The measure $dj_3(2j_3 + 1)^2$ is just the Plancherel-measure that appears in the harmonic analysis on H_3^+ [21],[8].

By proceeding as in Sect.3 one now confirms that the leading term in the OPE is given by (65). The operators Φ^j are related to the operators Θ^j by $\Phi^j(x|z) = B(j)\Theta^j(x|z)$.

6. APPENDIX B: CURRENT ALGEBRA CONSTRAINTS ON OPE AND FOUR POINT FUNCTION

For both the discussion of the OPE of two generic primary fields and the description of the four-point function one needed to have some information on the nature of the descendant contributions in OPE and factorization. A more precise description of these contributions is made possible by means of the following useful technical tool:

6.1. Dual basis.

6.1.1. To begin with, one needs to introduce some notation. Given any basis $\{v_s; s \in \mathcal{S}\}$ for the zero mode subspace P_j one may introduce an useful basis for \mathcal{R}_j as follows: Let $\mathcal{T}(n)$ be the set of all tuples

$$(69) \quad \mu = ((r_1, \dots, r_i, \dots), (s_1, \dots, s_j, \dots), (t_1, \dots, t_k, \dots))$$

such that

$$n = n(\mu) \equiv \sum_{i=1}^{\infty} i r_i + \sum_{j=1}^{\infty} j s_j + \sum_{k=1}^{\infty} k t_k < \infty$$

Then the set of all

$$(70) \quad \mathfrak{J}_\mu v_s \equiv \prod_{i=1}^{\infty} (J_{-i}^+)^{r_i} \prod_{j=1}^{\infty} (J_{-j}^0)^{s_j} \prod_{k=1}^{\infty} (J_{-k}^-)^{t_k} v_s,$$

where $\mu \in \mathcal{T}(n)$, $n \in \mathbb{N}$ and $s \in \mathcal{S}$ forms a basis for \mathcal{R}_j .

6.1.2. The main result of this subsection is the construction of “dual” generators $\mathfrak{J}^\mu \in \mathcal{U}(\widehat{\mathfrak{sl}}_2)$ that are specified by

$$(71) \quad \Pi_0(\mathfrak{J}^\mu \mathfrak{J}_\nu) = \delta^\mu_\nu,$$

where Π_0 denotes projection onto the part that does not contain terms of the form $(\dots) J_n^a$ with $n > 0$. These “dual” generators will be of the form

$$(72) \quad \mathfrak{J}^\mu = \sum_{\nu \in \mathcal{T}(n(\mu))} \mathfrak{R}^{\mu\nu}(H, Q) \mathfrak{J}_\nu^\tau (J_0^{-\sigma(\mu, \nu)})^{\delta(\mu, \nu)},$$

where the conjugation τ is defined by $(J_n^a)^\tau = J_{-n}^{-a}$ and $(AB)^\tau = B^\tau A^\tau$, $\sigma(\mu, \nu) = \text{sgn}(m(\mu) - m(\nu))$, $\delta(\mu, \nu) = |m(\mu) - m(\nu)|$, $m(\mu) = \sum_{i=1}^{\infty} (r_i - t_i)$, $H = J_0^0$ and Q is the Casimir of the zero mode subalgebra, $Q = J_0^+ J_0^- - J_0^0(J_0^0 - 1)$.

The crucial point about the matrix $\mathfrak{R}^{\mu\nu}(H, Q)$ is the fact that it depends *polynomially* on the variable $H = J_0^0$ and rationally on the variable Q , with poles at $j = 2j_{r,s}^\pm + 1 = \pm(r + b^{-2}s)$ when $Q = j(j+1)$.

This result gives a convenient representation of the identity on \mathcal{V} as

$$(73) \quad \text{Id} = \int_{\mathcal{C}^+} dj \sum_{\mu, \bar{\mu}} \int_{\mathbb{C}} d^2x \mathfrak{J}_\mu \bar{\mathfrak{J}}_{\bar{\mu}} |j; x\rangle \langle j; x| \mathfrak{J}^\mu \bar{\mathfrak{J}}^{\bar{\mu}}.$$

6.1.3. The rest of this subsection will be devoted to the proof of this claim. In order to construct the \mathfrak{J}^μ one may consider the generators \mathfrak{J}_μ to be represented in Verma modules \mathcal{V}_j over the current algebra: The definition (71) is equivalent to validity of

$$(74) \quad \mathfrak{J}^\mu \mathfrak{J}_\nu f_0 = \delta^\mu_\nu f_0,$$

for any $f_0 \in \mathcal{V}_j$ such that $J_n^a f_0 = 0$ and generic j . The existence of “dual” generators \mathfrak{J}^μ for generic j will then follow from the Kac-Kazhdan determinant formula [23]. What is not evident is the claim on the polynomial dependence of the \mathfrak{J}^μ on the generator H .

To control this dependence it is useful to choose as basis for the Verma module the set

$$(75) \quad \{e_{m,\nu}^j \equiv \mathfrak{J}_\nu e_{m-m(\nu)}^j; \nu \in \mathcal{T}; m = j, j-1, \dots\},$$

where the e_m^j form a basis for the zero mode subspace $\mathcal{V}_{j,0} = \{f_0 \in \mathcal{V}_j : J_n^a f_0 = 0\}$ such that

$$(76) \quad J_0^+ e_m^j = -(m-j) e_{m+1}^j \quad J_0^- e_m^j = -(m+j) e_{m-1}^j \quad H e_m^j = m e_m^j.$$

One may then introduce the bilinear form (\cdot, \cdot) by

$$(77) \quad (\mathfrak{J}_\mu e_m^j, \mathfrak{J}_\nu e_n^j) = (e_m^j, \mathfrak{J}_\mu^\tau \mathfrak{J}_\nu e_n^j) \quad (e_m^j, e_n^j) = \delta_{m,n}.$$

This bilinear form is *not* the usual Shapovalov form $\langle \cdot, \cdot \rangle$ considered in [23], but closely related to it:

$$(78) \quad \langle e_m^j, e_n^j \rangle (\mathfrak{J}_\mu e_m^j, \mathfrak{J}_\nu e_n^j) = \langle \mathfrak{J}_\mu e_m^j, \mathfrak{J}_\nu e_n^j \rangle (e_m^j, e_n^j).$$

It follows that the determinant D_n^j of the matrix $B_{\mu\nu} = (e_{m,\mu}^j, e_{m,\nu}^j)$, with $\mu, \nu \in \mathcal{T}(n)$ is proportional to the Kac-Kazhdan determinant:

$$(79) \quad D_n^j = N_n (k-2)^{R_3(n)} \prod_{r,s \geq 1} \left((j - j_{r,s}^+) (j - j_{r,s}^-) \right)^{P_3(n-rs)},$$

where $2j_{r,s}^\pm + 1 = \pm(r + b^{-2}s)$, $P_3(n)$ is the cardinality of $\mathcal{T}(n)$, $R_3(n) = \sum_{r,s \geq 1} P_3(n-rs)$. The prefactor N_n may be determined by considering the highest power in the central charge k , which is produced by the product of the diagonal elements of the matrix $B_{\mu\nu}$. The important point here is that it turns out to be m -independent for the chosen basis, which implies m -independence of D_n^j .

6.1.4. The inverse of the matrix $B_{\mu\nu}$ therefore exists for $j \neq j_{r,s}^\pm$. The matrix elements of this inverse will be denoted $B^{\mu\nu}$. It will be necessary to describe their m -dependence a bit more precisely:

Let $s = -\sigma(\mu, \nu)$, $I_{\mu\nu} = \{\delta(\nu), \delta(\nu) + s, \dots, \delta(\mu) - s\}$. $B^{\mu\nu}$ factorizes as

$$(80) \quad B^{\mu\nu} = C^{\mu\nu} \prod_{k \in I_{\mu\nu}} (m - sj - k),$$

where $C^{\mu\nu} = C^{\mu\nu}(m, j)$ is a polynomial in m .

This may be seen as follows: It suffices to discuss the case $m(\nu) > m(\mu)$, the other case being completely analogous. One may equivalently fix a value of k and ask in which elements $B^{\mu\nu}$ a factor $(m - j - k)$ can appear.

Now one may use the following simple fact: If for a matrix M_{ij} the upper right submatrix with lower left corner at $(k-1, k)$ carries a common factor a , then the same will be true

for the inverse matrix. This may be verified by e.g. using the representation of the elements M^{ij} of the inverse as $M^{ij} = (\det M)^{-1} \bar{M}^{ij}$ where \bar{M}^{ij} is the determinant of the matrix obtained from M by erasing the i^{th} row and j^{th} column. The contributions to these determinants may be associated to zig-zag paths that go in unit steps from the right to the left in the matrix \bar{M}^{ij} and which never hit the same vertical position twice. The matrix element M^{ij} will carry a factor a if there exists no path that avoids the upper right submatrix in \bar{M}^{ij} with lower left corner at $(k-1, k)$. This is in fact the case if $i < k \leq j$: The lower right submatrix in \bar{M}^{ij} with upper left corner (k, k) then has less columns than rows.

It therefore suffices to verify that the factor $(m-j-k)$ appears in any matrix element $B_{\mu\nu}$ with $m(\mu) < k \leq m(\nu)$. But this follows from the definition by observing that

$$\mathfrak{J}_\mu^\tau \mathfrak{J}_\nu e_{m-m(\nu)}^j = R_{\mu\nu}(H, Q) (J_0^+)^{m(\nu)-m(\mu)} e_{m-m(\nu)}^j,$$

where $R_{\mu\nu}(H, Q)$ is some polynomial in the indicated variables.

6.1.5. The factorization (80) now allows to rewrite the identity on the Verma module \mathcal{V}_j as follows:

$$\begin{aligned} \text{Id} &= \sum_{m=-\infty}^j \sum_{\mu, \nu} \mathfrak{J}_\mu e_{m-m(\mu)}^j B^{\mu\nu}(m, j) (\mathfrak{J}_\nu e_{m-m(\nu)}^j)^\tau \\ (81) \quad &= \sum_{m=-\infty}^j \sum_{\mu, \nu} \mathfrak{J}_\mu e_{m-m(\mu)}^j C^{\mu\nu}(m, j) ((J_0^{-\sigma(\mu, \nu)})^{\delta(\mu, \nu)} e_{m-m(\mu)}^j)^\tau \mathfrak{J}_\nu^\tau, \end{aligned}$$

where the summation over μ, ν is restricted by $m-m(\mu) \leq j$ and $m-m(\nu) \leq j$. The summations over μ, ν and m may be exchanged, giving a sum over m that is restricted by the conditions just mentioned. The summation over m may then be shifted by $m(\mu)$, where restricting the new sum by $m-m(\nu)+m(\mu) \leq j$ is superfluous since automatically realized due to $J^+ e_j^j = 0$. The result may be written as

$$(82) \quad \text{Id} = \sum_{\mu} \sum_{m=-\infty}^j \mathfrak{J}_\mu e_m^j (e_m^j)^\tau \mathfrak{J}^\mu,$$

which is equivalent to the desired equation (74) with dual generators \mathfrak{J}^μ being given by

$$(83) \quad \mathfrak{J}^\mu \equiv \sum_{\nu} C^{\mu\nu}(H+m(\mu), Q) \mathfrak{J}_\nu^\tau (J_0^{\sigma(\mu, \nu)})^{\delta(\mu, \nu)}$$

6.2. Descendant operators.

6.2.1. The descendants of the states $|j; x\rangle$ with fixed j represent elements of the dual \mathcal{R}_j^t of \mathcal{R}_j . The aim will be to associate an operator $\Phi^j(\mathbf{v}|z)$ to each distribution $\mathbf{v} \in \mathcal{R}_j^t$. This can be done by defining recursively

$$\begin{aligned} \Phi^j(J_{-n}^a \mathbf{v}|z) &= \frac{1}{(n-1)!} : \partial_z^{n-1} J^a(z) \Phi^j(\mathbf{v}|z) :, \\ (84) \quad \Phi^j(\delta_x|z) &\equiv \Phi^j(x|z) \\ \Phi^j(\bar{J}_{-n}^a \mathbf{v}|z) &= \frac{1}{(n-1)!} : \partial_{\bar{z}}^{n-1} \bar{J}^a(z) \Phi^j(\mathbf{v}|z) :, \end{aligned}$$

where $\delta_x \in P_j^t$ represents the delta-functional $\delta_x(f) = f(x)$ for any $f \in P_j \simeq \mathcal{S}(\mathbb{C})$. Normal ordering is defined here by $: J^a(z) \mathcal{O}(z) := J_{<}^a(z) \mathcal{O}(z) + \mathcal{O}(z) J_{>}^a(z)$ where

$$J_{<}^a(z) = \sum_{n < 0} J_n^a z^{-n-1}, \quad J_{>}^a(z) = \sum_{n \geq 0} J_n^a z^{-n-1}.$$

State-operator correspondence (15) generalizes to

$$(85) \quad \lim_{z \rightarrow 0} \Phi^j(\mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} \delta_x | z) | 0 \rangle = \mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} | j; x \rangle \quad \text{or} \quad e^{zL-1} \mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} | j; x \rangle = \Phi^j(\mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} \delta_x | z) | 0 \rangle.$$

The transformation properties of descendants under the current algebra are given by the commutation relations

$$(86) \quad [J_{>}^a(w), \Phi^j(\mathfrak{v} | z)] = \Phi^j(J_{>}^a(w - z) \mathfrak{v} | z) \quad [J_{<}^a(w), \Phi^j(\mathfrak{v} | z)] = -\Phi^j(J_{>}^a(w - z) \mathfrak{v} | z).$$

One should note that $J_0^a \delta_x = \mathcal{D}_j^a \delta_x$, so that (86) includes the transformation law (11) as a special case.

6.2.2. The conformal Ward identities are now a consequence of the definition (84), commutation relations (86) and invariance of the vacuum $|0\rangle$: One may express any correlation function of descendant operators $\langle \Phi^{j_N}(\mathfrak{v}_N | z_N) \dots \Phi^{j_1}(\mathfrak{v}_1 | z_1) \rangle$ in terms of $\langle \Phi^{j_N}(x_N | z_N) \dots \Phi^{j_1}(x_1 | z_1) \rangle$ by expressing any chosen $\Phi^{j_i}(\mathfrak{v}_i | z_i)$ in terms of $\Phi^{j_i}(x_i | z_i)$ with the help of (84), moving the $J_{<}^a$, $\bar{J}_{<}^a$, (resp. $J_{>}^a$, $\bar{J}_{>}^a$) to the left (resp. right) vacuum by using (86), and then repeating the process until only primary fields are left.

In the case of the two point function of two descendant operators one needs to supplement these definitions by requiring it to vanish when the levels of the descendants of the two operators are unequal.

$$(87) \quad \langle \Phi^{-j_2-1}(\mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} \delta_{x_2} | z_2) \Phi^{j_1}(\mathfrak{J}_\nu \tilde{\mathfrak{J}}_{\bar{\nu}} \delta_{x_1} | z_1) \rangle = \\ = |z_{21}|^{-4\Delta_{j_1}} z_{21}^{-2n(\mu)} z_{21}^{-2n(\bar{\mu})} \delta_{n(\mu), n(\nu)} \delta_{n(\bar{\nu}), n(\bar{\mu})} \langle j_2; x_2 | \mathfrak{J}_\mu^\sigma \tilde{\mathfrak{J}}_{\bar{\mu}}^\sigma \mathfrak{J}_\nu \tilde{\mathfrak{J}}_{\bar{\nu}} | j_1; x_1 \rangle,$$

where the conjugation σ acts as $(J_n^a)^\sigma = J_{-n}^a$.

It follows from (87) that out-states are created via

$$(88) \quad \langle j; x | (\mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}})^\sigma = \lim_{z \rightarrow \infty} |z|^{4\Delta_{j_1}} z^{2n(\mu)} z^{2n(\bar{\mu})} \langle 0 | \Phi^j(\mathfrak{J}_\mu \tilde{\mathfrak{J}}_{\bar{\mu}} \delta_x | z)$$

6.3. Descendant contributions in OPE and factorization.

6.3.1. The resolution of the identity (73) immediately gives an expansion for the “in”-state $\Phi^{j_2}(x_2 | z_2) | j_1; x_1 \rangle$ with coefficients given by the matrix elements $\langle j; x | \mathfrak{J}^\mu \tilde{\mathfrak{J}}^{\bar{\mu}} \Phi^{j_2}(x_2 | z_2) | j_1; x_1 \rangle$. This is brought into correspondence with (27) by using the $j_3 \rightarrow -j_3 - 1$ symmetry of the integrand (cf. 3.1.1). One gets an expansion of the form

$$(89) \quad \Phi^{j_2}(x_2 | z_2) | j_1; x_1 \rangle = \\ \int_{\mathbb{C}^+} dj_3 \sum_{\mu, \bar{\mu}} D(j_3, j_2, j_1) z^{\Delta_{21} + n(\mu)} \bar{z}^{\Delta_{21} + n(\bar{\mu})} \cdot \\ \cdot \int_{\mathbb{C}} d^2 x_3 (\mathcal{D}_{x_2}^{j_2}(\mathfrak{J}^\mu) \mathcal{D}_{\bar{x}_2}^{j_2}(\tilde{\mathfrak{J}}^{\bar{\mu}}) C(j_3, j_2, j_1 | x_3, x_2, x_1)) | -j_3 - 1, x_3 \rangle,$$

where the differential operators $\mathcal{D}_{x_2}^{j_2}(\mathfrak{J}^\mu)$ are obtained by replacing in the expression for the dual generators $\mathfrak{J}^\mu(j_3)$ all generators J_n^a that appear by the differential operators $\mathcal{D}_{x_2}^a(j_2)$.

6.3.2. The full operator product expansion is now easily obtained by applying the translation operator $e^{z_1 L_{-1}}$ to (89) and using (85):

$$\begin{aligned}
 (90) \quad & \Phi^{j_2}(x_2|z_2)\Phi^{j_1}(x_1|z_1) = \\
 & = \int_{-\frac{1}{2}+i\mathbb{R}^+} dj_3 \, D(j_3, j_2, j_1) \sum_{\mu, \bar{\mu}} |z_{21}|^{2\Delta_{21}(j_3)} z_{21}^{n(\mu)} \bar{z}_{21}^{n(\bar{\mu})} \cdot \\
 & \cdot \int_{\mathbb{C}} d^2 x_3 \left(\mathcal{D}_{x_2}^{j_2}(\mathfrak{J}^\mu) \mathcal{D}_{\bar{x}_2}^{j_2}(\mathfrak{J}^{\bar{\mu}}) C(j_3, j_2, j_1|x_3, x_2, x_1) \right) \Phi^{j_3}(\mathfrak{J}_\mu \bar{\mathfrak{J}}_{\bar{\mu}} \delta_{x_3}|z_1).
 \end{aligned}$$

6.3.3. It is worth noting that the polynomial dependence of the \mathfrak{J}^μ on the zero mode generators J_0^a is of course crucial for getting differential operators of finite order in (90).

It also follows that the descendant contributions do not introduce any j_2, j_1 -dependent pole of the OPE-coefficients in addition to the poles discussed in Section 3.1.

6.3.4. *Four point function.* Projective invariance determines the z -dependence of the four point function to be of the form

$$\begin{aligned}
 (91) \quad & \langle \Phi^{j_4}(x_4|z_4) \dots \Phi^{j_1}(x_1|z_1) \rangle = |z_{43}|^{2(\Delta_2+\Delta_1-\Delta_4-\Delta_3)} |z_{42}|^{-4\Delta_2} |z_{41}|^{2(\Delta_3+\Delta_2-\Delta_4-\Delta_1)} \cdot \\
 & \cdot |z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)} \langle j_4; x_4 | \Phi^{j_3}(x_3|1) \Phi^{j_2}(x_2|z) | j_1; x_1 \rangle,
 \end{aligned}$$

where z is the usual cross ratio defined below (45). In order to get information on the $F_j(J|X|z)$ is therefore suffices to consider the case $z_4 = \infty$, $z_3 = 1$, $z_2 = z$, $z_1 = 0$.

A representation of the four point function as sum over intermediate states can be obtained e.g. by expanding $\Phi^{j_2}(x_2|z)|j_1; x_1\rangle$ as in (89), using (73) to get a similar expansion for the “out”-state $\langle j_4; x_4 | \Phi^{j_3}(x_3|1)$ and finally using the inner products (7). One obtains a representation of the four point function as sum over products of three point functions which takes the form

$$\begin{aligned}
 (92) \quad & \langle \Phi^{j_4} \dots \Phi^{j_1} \rangle = \int_{-\frac{1}{2}+i\mathbb{R}^+} dj \sum_{\mu, \bar{\mu}} z^{\Delta_{21}+n(\mu)} \bar{z}^{\Delta_{21}+n(\bar{\mu})} D_{43}(j) B(-j-1) D_{21}(j) \cdot \\
 & \cdot \tilde{\mathcal{D}}_{x_3}^{j_3}(\mathfrak{J}_\mu) \tilde{\mathcal{D}}_{\bar{x}_3}^{j_3}(\bar{\mathfrak{J}}_{\bar{\mu}}) \mathcal{D}_{x_2}^{j_2}(\mathfrak{J}^\mu) \mathcal{D}_{\bar{x}_2}^{j_2}(\bar{\mathfrak{J}}^{\bar{\mu}}) G_j(J|X),
 \end{aligned}$$

where the function $G_j(J|X)$ is given by the integral (46) and the differential operators $\tilde{\mathcal{D}}_x^j(\mathfrak{J}_\mu)$ are obtained by replacing any generator J_n^a that appears in \mathfrak{J}_μ by the differential operator $-\mathcal{D}_{j,x}^a$.

The differential operators $\mathcal{D}_j^{(n)}(J|X)$ that appear in (47) are the finally defined by summing the differential operators appearing in (92) over μ with $n(\mu) = n$.

7. APPENDIX C: KNIZHNIK-ZAMOLODCHIKOV EQUATIONS

The derivation of the KZ-equations for the H_3^+ -WZNW model is very similar to the case of WZNW-models for compact groups, see e.g. [24] for a derivation that can be easily

adapted to the formalism used in Appendix B. It is based on the result that the operators satisfy the following operator differential equation in the sense of formal power series:

$$(93) \quad \frac{d}{dz} \Phi^j(\delta_x|z) = \frac{1}{2(k-2)} \sum_{a,b=+,0,-} \eta_{ab} \Phi^j(J_{-1}^a J_0^b \delta_x|z),$$

where $\eta_{00} = -2$, $\eta_{+-} = 1 = \eta_{-+}$ are the only nonvanishing matrix elements of η_{ab} .

When inserting (93) into correlation functions and using the conformal Ward identities one finds the KZ-equations (53).

7.1. Solutions to KZ. Solutions that can be identified with s-channel conformal blocks must be of the form :

$$F_j(J|x|z) = z^{\Delta_{21}(j)} \sum_{n=0}^{\infty} z^n F_j^{(n)}(J|x)$$

The KZ-equation (55) then implies the following set of recursion relations for the coefficients $F_j^{(n)}(J|x)$:

$$(94) \quad (\Delta_{21} + n + (k-2)^{-1}P) F_j^{(n)}(J|x) = -\frac{1}{k-2} Q \sum_{m=0}^{n-1} F_j^{(m)}(J|x)$$

For $n = 0$ one has the eigenvalue equation $(P + (k-2)\Delta_{21}) F_j^{(0)}(J|x) = 0$. Two linearly independent solutions are $F_j(J|x)$ and $F_{-j-1}(J|x)$, cf. (50).

The equations with $n > 0$ are inhomogeneous differential equations that uniquely determine $F_j^{(n)}(J|x)$ in terms of $F_j^{(m)}(J|x)$, $m < n$ up to solutions of the homogeneous equation $(\Delta_{21} + n + (k-2)^{-1}P)f = 0$. However, the corresponding ambiguity can be fixed by observing that on the one hand solutions which can be identified with conformal blocks have to be of the form

$$(95) \quad F_j^{(n)}(J|x) = x^{j_1+j_2-j-\kappa(n)} \sum_{m=0}^{\infty} x^m F_j^{(n,m)}(J),$$

where $\kappa(n)$ is integer as follows from (94). But on the other hand one may observe that the exponents λ_n of solutions to the homogeneous equation $(\Delta_{21} + n + (k-2)^{-1}P)f = 0$ at the singular point $x = 0$ are determined by the equation

$$(\lambda_n - (j_1 + j_2 + \frac{1}{2}))^2 = (j + \frac{1}{2})^2 - (k-2)n,$$

so if one writes $\lambda_n = j_1 + j_2 - j - l(n)$ one finds that $(2j+1)l(n) = (k-2)n$. Unless $2j+1$ is a rational multiple of $k-2$ one finds that $l(n)$ is not an integer so that adding a solution to the homogeneous equation to a solution of (94) would prevent the identification of the solution with a conformal block.

Furthermore one has to check that there always exists a solution of the required form (95). First observe that In order to satisfy (94) the coefficients $F_j^{(n,m)}(J)$ will have to satisfy recursion relations of the following form:

$$A_m F_j^{(m,n)} + B_m F_j^{(n,m-1)}(J) = G_j^{(n,m-1)}(J)$$

where the coefficients are given by

$$\begin{aligned} A_m &= \left(\tau_n + m - \left(j_1 + j_2 + \frac{1}{2} \right) \right)^2 - \left(j_{21} + \frac{1}{2} \right)^2 + (k-2)n ; \\ B_m &= (\tau_n + m - \kappa)(\tau_n + m - 2j_2) \end{aligned} \quad ; \quad \tau_n \equiv j_1 + j_2 - j_{21} - n$$

and the $G_j^{(n,m-1)}(J)$ are defined by

$$x^{\tau_n} \sum_{m=0}^{\infty} x^m G_j^{(n,m-1)}(J) \equiv Q \sum_{i=0}^{n-1} F_{j_{21},n}^{(s)}(x)$$

The recursion relations are solvable unless there is some m for which $A_m = 0$. But $A_m = (m-n)^2 - (2j_{21}+1)(m-n) + (k-2)n$, which will not vanish for any integer m unless $2j+1$ is a rational multiple of $k-2$.

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SCHOOL FOR THEORETICAL PHYSICS, DUBLIN INSTITUTE FOR ADVANCED STUDIES, 10 BURLINGTON ROAD, DUBLIN 4, IRELAND

E-mail address: `teschner@stp.dias.ie`